

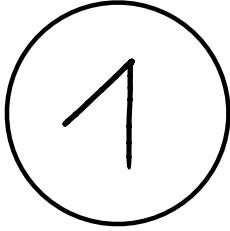
A new approach to Odd Khovanov Homology

slides on my website!

December 5, 2025
Berlin-Brandenburg workshop

Léo Schelstraete
Max Planck Institute for Mathematics (Bonn)

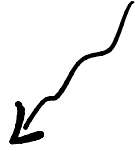




What is...
odd Khovanov homology?

L

$\{\text{links in } \mathbb{R}^3\}$



$Kh_{i,j}(L)$

(even) Khovanov
homology

$\{\text{bigraded abelian groups}\}$

•

•

•

homological

quantum

$Kh_{i,j}(L)$
(even) Khovanov
homology

L

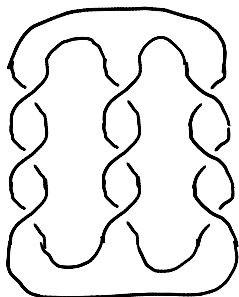
$OKh_{i,j}(L)$
odd Khovanov
homology

{links in \mathbb{R}^3 }

{bigraded abelian
groups}

- $Jones(L) = \chi_q(Kh(L)) = \chi_q(OKh(L))$
- same skein exact sequence!
- $Kh(L, \mathbb{F}_2) \cong OKh(L, \mathbb{F}_2)$





$P(3,3,-3)$

even

| | | | | | | | | |
|-----|---|----|----|----|----|----|----|---|
| | | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
| 0 | | | | | | | | 2 |
| -2 | | | | | | | 1 | |
| -4 | | | | | | 1 | | |
| -6 | | | | 2 | | | | |
| -8 | | | 1 | | | | | |
| -10 | | 1 | | | | | | |
| -12 | 1 | | | | | | | |

odd

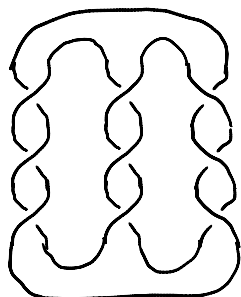
| | | | | | | | | |
|-----|---|----|----|----|----|----|----|----------------|
| | | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
| 0 | | | | | | | | 2 |
| -2 | | | | | | | 1 | 1 ₃ |
| -4 | | | | | | 1 | | |
| -6 | | | | 2 | | | | |
| -8 | | | 1 | | | | | |
| -10 | | 1 | | | | | | |
| -12 | 1 | | | | | | | |

[Shumakovitch 11]

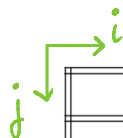
Theorem (Ebert–S. 25).

Theorem (Migdal–Wehrli 24).

Conjecture (Oszvath–Rasmussen–Szabó 07).



$P(3,3,-3)$



even

| | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
|-----|----|----|----|----|----|----|---|
| 0 | | | | | | | 2 |
| -2 | | | | | | 1 | |
| -4 | | | | | 1 | | |
| -6 | | | | 2 | | | |
| -8 | | | 1 | | | | |
| -10 | | 1 | | | | | |
| -12 | 1 | | | | | | |

odd

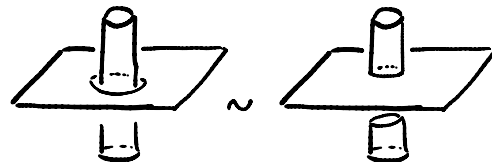
| | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
|-----|----|----|----|----|----|----|----------------|
| 0 | | | | | | | 2 |
| -2 | | | | | | 1 | 1 ₃ |
| -4 | | | | | 1 | | |
| -6 | | | | 2 | | | |
| -8 | | | 1 | | | | |
| -10 | | 1 | | | | | |
| -12 | 1 | | | | | | |

[Shumakovitch 11]

3-torsion!

Theorem (Ebert–S. 25). $OKh(P(n,n,-n))$ has n -torsion.

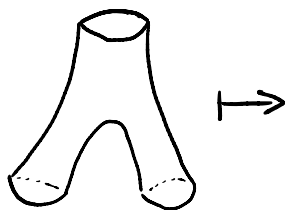
Theorem (Migdal–Wehrli 24). Odd Khovanov homology detects ribbon moves.



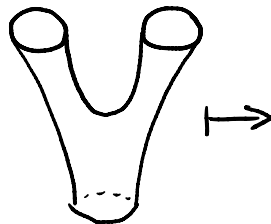
Conjecture (Oszváth–Rasmussen–Szabó 07). $OKh(L) \Rightarrow \widehat{HF}(\Sigma(L))$

even

odd



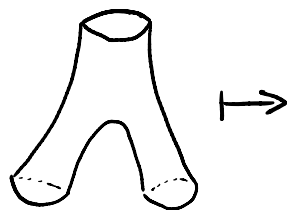
$$\begin{array}{c} \mathbb{Z}[\kappa]/\kappa^2 \\ \uparrow \textcolor{brown}{P} \mid x_1, x_2 \mapsto \kappa \\ \mathbb{Z}[x_1, x_2]/x_1^2, x_2^2 \end{array}$$



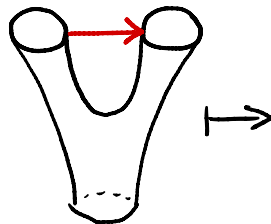
$$\begin{array}{c} \mathbb{Z}[x_1, x_2]/x_1^2, x_2^2 \\ \uparrow (\kappa_1 + \kappa_2) \textcolor{brown}{P} \mid x \mapsto \kappa_1 \\ \mathbb{Z}[\kappa]/\kappa^2 \end{array}$$

even

odd



$$\begin{array}{c} \mathbb{Z}[\kappa]/\kappa^2 \\ \uparrow \textcolor{brown}{P} |_{x_1, x_2 \mapsto \kappa} \\ \mathbb{Z}[x_1, x_2] / x_1^2, x_2^2 \end{array}$$

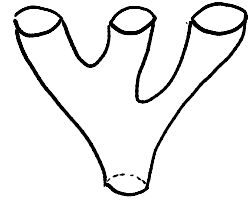
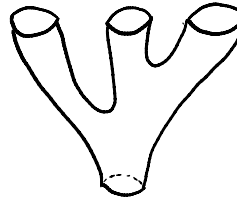
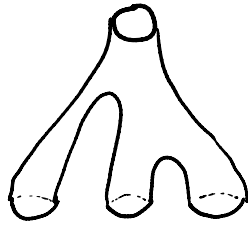
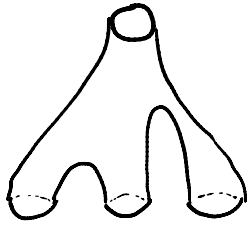


$$\begin{array}{c} \mathbb{Z}[x_1, x_2] / x_1^2, x_2^2 \\ \uparrow (x_1 + x_2) \textcolor{brown}{P} |_{x \mapsto x_1} \\ \mathbb{Z}[\kappa]/\kappa^2 \end{array}$$

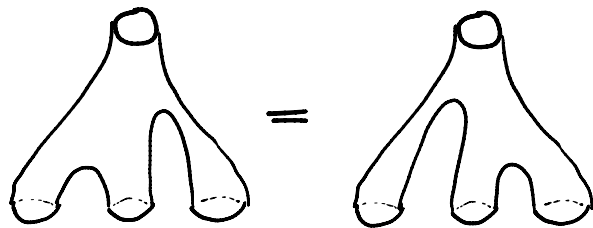
$$\begin{array}{c} \Lambda(x) \\ \uparrow \textcolor{brown}{P} |_{x_1, x_2 \mapsto x} \\ \Lambda(x_1, x_2) \end{array}$$

$$\begin{array}{c} \Lambda(x_1, x_2) \\ \uparrow (x_1 - x_2) \Lambda \textcolor{brown}{P} |_{x \mapsto x_1} \\ \Lambda(x) \end{array}$$

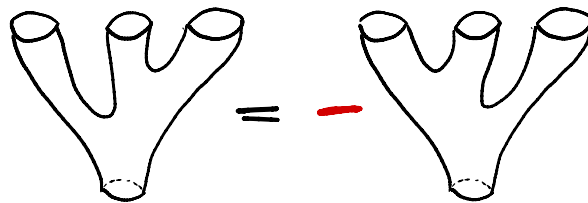
$$\Lambda(x_1, \dots, x_m) = \mathbb{Z}\langle x_1, \dots, x_m \rangle / (x_i x_j = -x_j x_i, x_i^2 = 0)$$



Theorem (Oszváth–Rasmussen–Szabó 07).

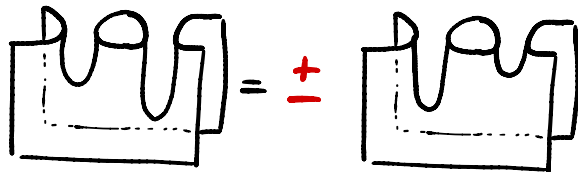


BUT



Theorem (Oszv th–Rasmussen–Szab  07). Still gives a link invariant!

Q1. Extension to tangles?



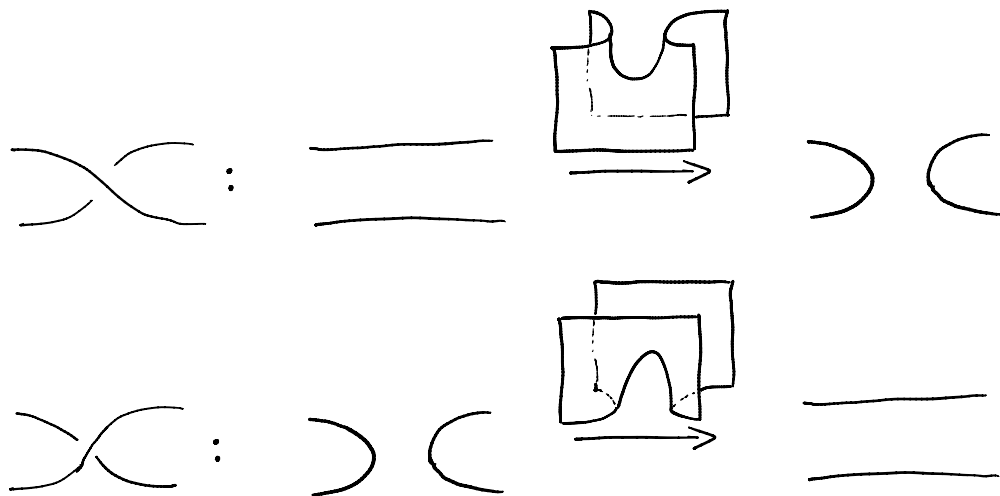
not local!

Q2. What is an “**odd**” link homology, *really*?

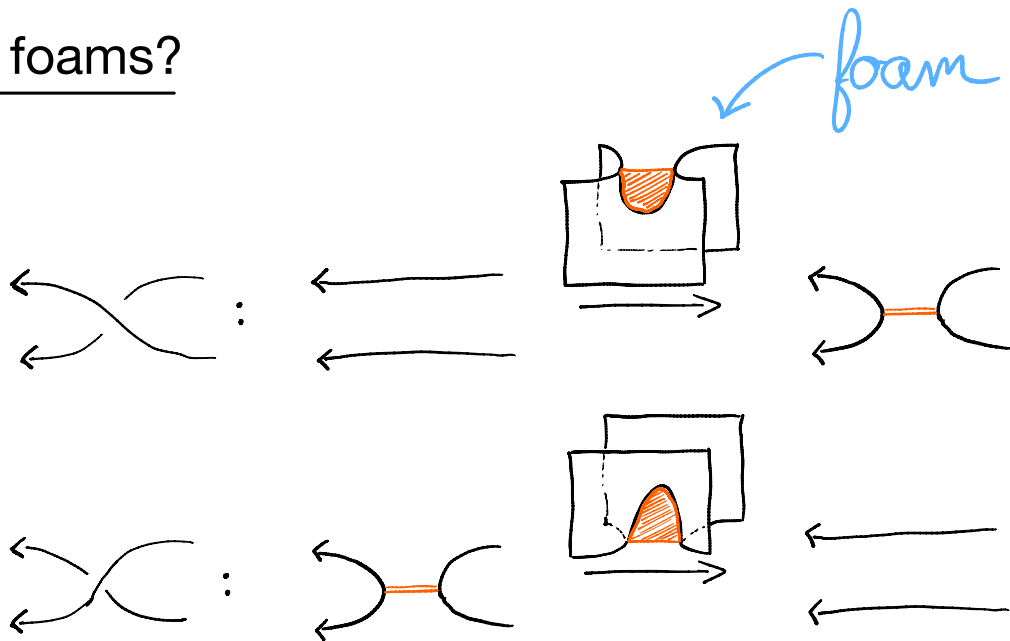
2

A novel approach: super foams

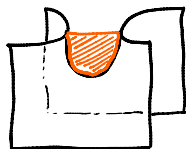
What are... foams?



What are... foams?



two saddles



even

and

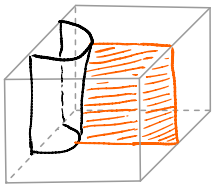
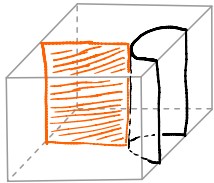
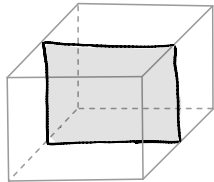
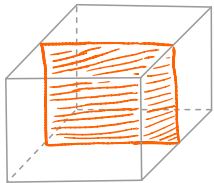
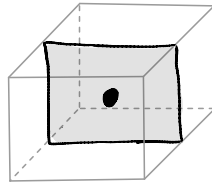
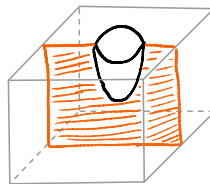
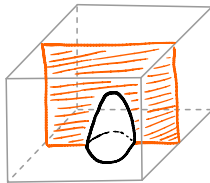
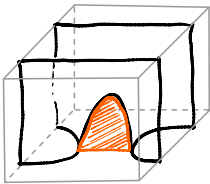
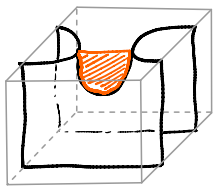
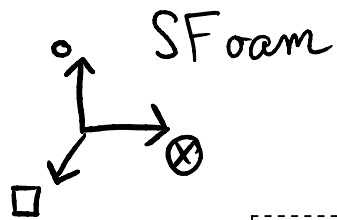


odd

\in

the monoidal
2-supercategory

\mathcal{SFoam}



$$SFoam = \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \end{array} \right\rangle / \sim$$

The image shows five 3D diagrams of a cube, each containing a different geometric feature:

- Diagram 1: A cube with a black wireframe. Inside, there is an orange-shaded, bowl-shaped surface (a 2D disk) and a black loop (a 1D circle) that passes through the center of the disk.
- Diagram 2: A cube with a black wireframe. Inside, there is an orange-shaded, bowl-shaped surface (a 2D disk) and a black loop (a 1D circle) that passes through the center of the disk.
- Diagram 3: A cube with a black wireframe. Inside, there is an orange-shaded rectangular prism (a 3D volume) and a black loop (a 1D circle) that passes through the center of the prism.
- Diagram 4: A cube with a black wireframe. Inside, there is an orange-shaded rectangular prism (a 3D volume) and a black loop (a 1D circle) that passes through the center of the prism.
- Diagram 5: A cube with a black wireframe. Inside, there is a gray-shaded square (a 2D disk) with a black dot (a 0D point) at its center.

Theorem (S. 25).

$$SFoam = \left\langle \begin{array}{c} \text{[cube with orange top face and black top boundary]} \\ \text{even} \end{array} \begin{array}{c} \text{[cube with orange bottom face and black bottom boundary]} \\ \text{odd} \end{array} \begin{array}{c} \text{[cube with orange front face and black front boundary]} \\ \text{even} \end{array} \begin{array}{c} \text{[cube with orange back face and black back boundary]} \\ \text{odd} \end{array} \begin{array}{c} \text{[cube with gray back face and black dot]} \\ \text{odd} \end{array} \right\rangle / \sim$$

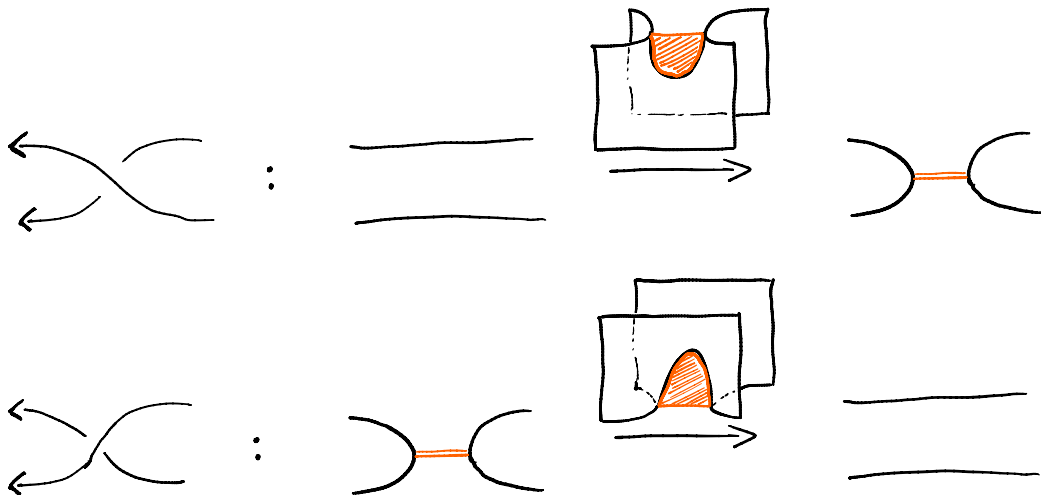
As a monoidal 2-**super**category! ("higher superalgebra")

$$\begin{array}{|c|c|} \hline \alpha & ID \\ \hline ID & \beta \\ \hline \end{array} = (-1)^{|\alpha||\beta|} \begin{array}{|c|c|} \hline ID & \alpha \\ \hline \beta & ID \\ \hline \end{array}$$

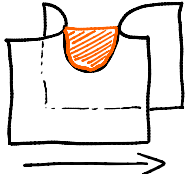
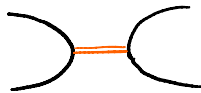
super interchange law

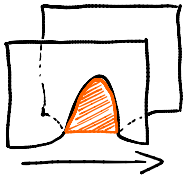

Theorem (S. 25). Homspaces in **SFoam** have the expected dimension.

Definition.




Definition. For β a braid, let $Ch(\beta) \in Ch(SFoam)$:

$$Ch\left(\begin{array}{c} \leftarrow \\ \searrow \\ \swarrow \\ \leftarrow \end{array}\right) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \xRightarrow{\text{diagram of a square with a shaded orange region}} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$



$$Ch\left(\begin{array}{c} \leftarrow \\ \swarrow \\ \searrow \\ \leftarrow \end{array}\right) = \begin{array}{c} \text{---} \\ \text{---} \end{array} \xRightarrow{\text{diagram of a square with a shaded orange region}} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$



and $Ch(\beta_1 \otimes \dots \otimes \beta_m) = Ch(\beta_1) \otimes \dots \otimes Ch(\beta_m).$

super \otimes of
chain complexes (S. 20) 

Theorem (S.-Vaz 23).

Theorem (S.-Vaz 23).

This recovers odd Khovanov homology

SLOGAN. Odd Khovanov homology
arises from the super interaction of
even $\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)$ and odd $\left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right)$ chain complexes.